准确封闭形式表达式：用于表征UWB光学系统的时宽乘积和FCC违规率的新指标

Eslam A. Aly, Mostafa B. Alabd, Nouran Kassem, Waleed Mustafa, Mohamed Shehata, Hassan Mostafa

Electronics and Communications Engineering Department, Cairo University, Giza 12613, Egypt

University of Science and Technology, Nanotechnology and Nanoelectronics Program, Zewail City of Science and Technology, October Gardens, 6th of October, Giza 12578, Egypt

Communications and Information Engineering Department at Zewail City for Science and Technology, Cairo, Egypt

ARTICLE INFO

Keywords:
UWB
TBP
FCC
Gaussian Pulse
Sech Pulse

ABSTRACT

在这篇论文中，对七种基本光学通讯系统中高斯和双曲正切脉冲的前七阶导数进行了分析。分析主要基于两个重要的设计指标，即联邦通信委员会违规率（FCC VR）和时宽乘积（TBP）。FCC VR是一个新指标，它计算信号相对于FCC掩模谱的平均功率的违规率。第二个讨论的指标是TBP，它基于脉冲宽度的方差定义获得。这个定义在许多方面比其他定义更有优势，如论文中所讨论的。在本工作中，获得了对上述脉冲形状的TBP的通用封闭形式表达式。此外，对基本光学通讯系统的分析表明，这两个指标对设计类似系统具有重要意义。

1. Introduction

无线和RF服务已经成为现代日常生活的必需品，新的技术和进步不断涌现。然而，伟大的进步总是伴随着限制。当前RF服务的频谱拥堵是其中最主要的挑战之一。在没有干扰到现有服务的情况下，很难为新技术和服务找到频谱。因此，联邦通信委员会（FCC）管理着所有现有和新RF服务的频谱。此外，频谱拥堵问题仍然是一个主要问题，由于用户对信息的范围和安全性越来越大，以及对任何时间、任何地点的需求不断增加[1][2]。

因此，一个主要为短程通信提出的设想，是超宽带（UWB），这是一种基于无线电能量在很大频率带宽上的传播的技术，具有很低的功率谱。
density; having the maximum PSD as -41.3 dBm in the range of 3.1 GHz to 10.6 GHz, that is known as the useful ultra-wide band. The low power spectral density limits the interference with the traditional RF services occupying the spectrum, and the large bandwidth allows for a very high data rate for the communications system[3]. In this paper, an analytic analysis of the first seven derivatives of the Gaussian and the hyperbolic secant pulses is carried out through a fundamental optical communications system, shown in Fig. 1. This analysis is based on two important metrics; the time-bandwidth product (TBP), and the FCC Violation ratio ratio (FCC VR), which is introduced in this work.

The paper is organized as follows; in section II, the system components and their role are discussed. In section III, the TBP is introduced and mathematically derived for the $m^{th}$-order derivative of the Gaussian and Sech pulses. In section IV, the FCC VR is also introduced and mathematically formulated. In section V, a general optical communications system is analyzed by applying the TBP and the FCC VR as design metrics, characterizing the design trade-offs present in a similar system.

2. System Overview

The system in in Fig. 1 consists of continuous wave (CW) laser diode (LD), followed by variable optical attenuator (VOA), which controls the optical power of laser diode output signal. Then, the Mach-Zhender modulator which is an electro-optic modulator used for amplitude modulation of optical signals. Finally, the amplified signal passes through the optical standard single-mode fiber (SMF) followed by the photo-detector, which is used to convert the received optical signal to electrical signal[3].

3. Time-Bandwidth product

The time-bandwidth product (TBP) is defined as the product of the temporal width and the spectral width of a pulse. In ultra-fast laser physics, it is common to define the width of a pulse as the full width at half-maximum (FWHM) in both time- and frequency-domain. However, in this paper, a different definition of the pulse width is applied for reasons to be discussed later in the section.

Transform-limited pulses (as opposed to chirped-pulses) have the minimum possible TBP. Therefore, TBP is often used for indicating how close a pulse is to the transform limit, by indicating how close the pulse duration is to the limit which is set by its spectral width.

3.1. The mathematical definition

The normalized $n^{th}$ moment functions in both time and frequency domains are first established, for a pulse described by $\Psi(X)$, as follows:

$$\langle X^n \rangle = \frac{\int_{-\infty}^{\infty} X^n |\Psi(X)|^2 dX}{\int_{-\infty}^{\infty} |\Psi(X)|^2 dX}$$  \hspace{1cm} (1)

Where $X$ is set as $t$ to donate time domain, and $\omega$ to donate frequency domain.

Furthermore, the temporal and spectral widths are defined based on the definition of the standard deviation as follows:

$$\Delta t = \sqrt{\langle t^2 \rangle - \langle t \rangle^2}$$ \hspace{1cm} (2)

$$\Delta \omega = \sqrt{\langle \omega^2 \rangle - \langle \omega \rangle^2}$$ \hspace{1cm} (3)

Thus, the TBP is defined in the following form:

$$\text{TBP} = \Delta \omega \cdot \Delta t$$ \hspace{1cm} (4)
The previous definition of the temporal width and the spectral width in (2) and (3) respectively has some advantages over the definition of the FWHM. Most importantly: its highly invariant mathematical definition that depends only on the pulse shape. Such a definition accurately characterizes odd functions such as the first derivative of the Gaussian pulse, unlike the FWHM. This proves very useful in the analysis carried out in this paper. In subsections B and C, the TBP for the Gaussian and hyperbolic secant pulses of general order derivatives are analytically derived. To analytically define the TBP of a pulse, the integrals from (1) (time- and frequency-domain moment functions) must be evaluated first.

3.2. The Gaussian pulse

Both temporal- and spectral-width derivations for the Gaussian pulse will be established in the following subsections to obtain a closed form for the TBP of the mth-order derivative of the Gaussian pulse.

3.2.1. The frequency domain

The following integral is to be evaluated to determine the spectral-width of the Gaussian pulse of general mth-order derivative:

\[ I = \frac{1}{\pi} \int_{-\infty}^{\infty} \omega (G^{(m)}(\omega))^2 d\omega \]  

(5)

Where the input basis function is the amplitude of the normalized Gaussian pulse, substituting in (5) leads to:

\[ I = \frac{1}{\pi} \int_{-\infty}^{\infty} \omega^{n+2m} e^{-\frac{\omega^2}{\tau^2}} d\omega \]  

(6)

This integral can be analytically evaluated to a closed formula through applying the repetitive differentiation of the Gaussian integral, which leads to (6) being analytically evaluated as the following expression:

\[ z(n, m) = \begin{cases} 
\frac{(2m + n - 1)!}{2} \sqrt{\pi} & \text{even}, \quad n \neq 0, \\
\frac{2^{m+0.5}n - 1.5 (n + 2m - 2)}{2^{2m+n-1}} & 2m + n \neq 0 \\
\sqrt{\frac{\pi}{\tau^2}} & 2m + n = 0 \\
0 & \text{otherwise} 
\end{cases} \]  

(7)

3.2.2. The time domain

Similarly, the following integral is to be evaluated to determine the temporal-width of the Gaussian pulse of general mth-order derivative:

\[ I = \int_{-\infty}^{\infty} t^n (g^{(m)}(t))^2 dt \]  

(8)

By substituting for the time domain expression for the Gaussian pulse, (8) leads to:

\[ I = \int_{-\infty}^{\infty} t^n ( - \frac{1}{\tau^2} )^{2m} e^{-2\frac{t^2}{\tau^2}} e^{\frac{t^2}{\tau^2}} H_{m}^{2} dt \]  

(9)

Where \( H_m \) is the Hermite Polynomial of order \( m \).

It is seen that squaring and multiplying the Hermite polynomial by \( t^n e^{-2t^2} \) would result in the sum of integrals similar to that of the frequency domain case (6), so (9) will be analytically evaluated as follows:

\[ I = \frac{1}{\pi^{n+1-2m}} (m!)^2 \sqrt{\pi} \left( z(n, m) + \sum_{j=0}^{m-1} \sum_{i=0}^{\frac{m-1-j}{2}} (-1)^{m-2-2(n-j)} (2m - 2i - 2j + n - 1)! \right) \]

(10)

Where \( z(n, m) \) is given by:

\[ z(n, m) = \begin{cases} 
\frac{(n + 1)!}{2^{n+0.5}(n-1)!} & \text{niseven, andmisodd} \\
\frac{2^{n+0.5}n - 1.5 (n - 1)!}{2^{2n-0.5}(n-1)!} & \text{niseven} \\
\frac{1}{2^{n+0.5}(n-1)!} & \text{andmiseven} \\
\sqrt{\pi} & n = 0 
\end{cases} \]  

(11)
By substituting for \(n\) and \(m\) in the equations from (7), (10), and (11); analytically determining the TBP for a general \(m\)th-order derivative Gaussian pulse is feasible.

### 3.3. The hyperbolic secant pulse

Both temporal- and spectral-width derivations for the hyperbolic secant pulse will be established in the following subsections to obtain a closed form for the TBP of the \(m\)th-order derivative of the hyperbolic secant pulse.

#### 3.3.1. The frequency domain

The following integral is to be evaluated to determine the spectral-width of the hyperbolic secant pulse of general \(m\)th-order derivative:

\[
I = \int_{-\infty}^{\infty} \omega^2 |S^{(m)}(\omega)|^2 d\omega
\]

The previous equation is then analytically evaluated as follows [4]:

\[
I = \frac{2}{\pi} \left( \frac{1}{2} \right)^{m+2m+1} B_{n+2m} \left( \frac{1}{2} \right)
\]

Where \(B_i(x_0)\) is the Bernoulli polynomial of order \(i\) substituted in with point \(x_0\).

#### 3.3.2. The time domain

Similarly, the following integral is to be evaluated to determine the temporal-width of the hyperbolic secant pulse of general \(m\)th-order derivative:

\[
I = \int_{-\infty}^{\infty} t^{m} |s^{(m)}(t)|^2 dt
\]

The previous equation is analytically evaluated in the following form:

\[
I = \left[ \sum_{r=0}^{\left\lfloor \frac{r}{2} \right\rfloor} \left( E_r^x \right)^2 \sum_{k=0}^{m-2r} \left( \frac{m - 2r}{k} \right) A_k \right] + 2 \sum_{r=0}^{\left\lfloor \frac{r}{2} \right\rfloor} \sum_{j=0}^{m-2r} E_r^x E_{m-2r} \left[ \sum_{k=0}^{m-2r} (-1)^j \left( \frac{m - r - j}{k} \right) A_k \right]
\]

Where \(A_k\) is defined as:

\[
A_k = (-1)^k \int_{-\infty}^{\infty} \text{sech}(t)^{2k+2} dt
\]

And where \(E_n^a\) is the double-indexed Euler number, defined for constants \(a\) and \(b\). The double-indexed Euler number is defined recursively by the following equation:

\[
E_n^a = E_{n-1}^{a-1}(m - 2r + 1) + E_{n-1}^{r}(n - 2m) \text{ and } E_n^a = m!
\]

The integral in (16) is then evaluated in Table 1 by taking the limits to the expression in (18) from [5].

\[
\left( \int x^n \text{sech}^n(x) dx \right) = -2^n e^{-nx} \left[ \frac{x^n}{n} f_1(n, \frac{n + 2}{2}, -e^{-2x}) \right] + kx^{k-1} f_2(n, \frac{n}{2}, \frac{n + 2}{2}, -e^{-2x}) + \frac{k(k - 1)x^{k-2}}{n^2} f_3(n, \frac{n}{2}, \frac{n + 2}{2}, -e^{-2x}) + \ldots + \frac{k!}{n^k + \ldots + k} f_{k+1}(n, \frac{n + 2}{2}; (k + 1)!) + C
\]

### 4. FCC Violation Ratio

The UWB technology was introduced as a technique to make better use of the RF spectrum without interfering with current RF services. UWB signals are short-range, carrier-less signals that can support very high bit rates.

As aforementioned, the FCC mask regulates the UWB technology with certain EIRP regulations over the 12GHz spectrum. However, generating UWB signals that perfectly follow the FCC mask is not feasible. Therefore, the FCC VR is introduced to quantify the violation of the generated pulses to the FCC constraints. It is defined as the average power of the pulse violating the FCC mask relative to the total average power of the FCC mask. FCC VR can be used as a design parameter to help achieve the highest possible
EIRP over the widest possible band while obeying the FCC regulations.

In this work, FCC VR is mathematically defined by first defining the row vector; which is a binary vector where the ones indicate the violation of the FCC mask as shown in the following equation, and in Fig. 2:

$$\rho = \frac{1}{4\pi BW} \int_{-\infty}^{\infty} \text{sign}(X_{\text{norm}}(\omega) - S_{\text{FCC}}(\omega)) + 1 \, d\omega$$

(19)

Where, $X_{\text{norm}}$ from (19) is the spectrum of the generated pulse normalized to the FCC mask spectral-limit.

As illustrated in Fig. 2, the shaded area is the area that violates the FCC mask, and it is represented by the one in the binary row vector. Thus, the FCC VR is mathematically defined as follows:

$$\text{FCCVR} = \frac{\int_{-\infty}^{\infty} \rho(\omega) \times X_{\text{norm}}(\omega) \, d\omega}{\int_{-\infty}^{\infty} S_{\text{FCC}}(\omega) \, d\omega}$$

(20)

Equation (20) indicates the compliance of the generated pulse to the FCC mask. Furthermore, Figs. 3 and 4 illustrate the change of

<table>
<thead>
<tr>
<th>$x^0$</th>
<th>$x^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>sech(x)$^2$</td>
<td>2</td>
</tr>
<tr>
<td>sech(x)$^4$</td>
<td>1.33</td>
</tr>
<tr>
<td>sech(x)$^6$</td>
<td>1.07</td>
</tr>
<tr>
<td>sech(x)$^8$</td>
<td>0.91</td>
</tr>
<tr>
<td>sech(x)$^{10}$</td>
<td>0.81</td>
</tr>
<tr>
<td>sech(x)$^{12}$</td>
<td>0.73</td>
</tr>
<tr>
<td>sech(x)$^{14}$</td>
<td>0.68</td>
</tr>
<tr>
<td>sech(x)$^{16}$</td>
<td>0.64</td>
</tr>
</tbody>
</table>

Table 1
Results for $\int_{-\infty}^{\infty} \text{sech}(x)^{2k+2} \, dx$

Fig. 2. Row vector for the first derivative of the Gaussian pulse.
the FCC VR with the full-width at half maximum (FWHM) for the first seven derivatives of the Gaussian and hyperbolic secant pulses respectively. As illustrated in Figs. 3 and 4, higher-order derivatives of the Gaussian- and Sech-shaped pulses reach lower FCC VR over a wider range of the FWHM.

5. The optical communications system

In this section, analysis of the basic optical communications system Fig. 1 is carried out. Furthermore, the TBP and FCC VR are applied throughout the system and proven to be very useful design parameters for a similar system.

Calculating the TBP plays two important roles in this system; it indicates the effect of the modulation on the transmitted pulse being close to its transform limit, also it provides the limitation on the design of the optical fiber as will be seen in the following section.

This work is mainly concerned with studying the communication channel (optical fiber) of the optical system Fig. 1, and applying the discussed design metrics, namely: TBP and FCC VR.

5.1. The optical fiber

Optical fibers have become a core technology in long-distance telecommunications and LANs, allowing extremely fast and low-
cost transmission of mostly digital data. In this work, the communication channel under analysis is a single mode optical fiber (SMF).

The propagation in the SMF is characterized by the wave equation, which is then transformed to the generalized nonlinear Schrödinger equation (GNLSE) [6]. The GNLSE can be numerically solved using the split-step Fourier method (SSFM) [7], which is a pseudo spectral numerical method that is usually applied to solve nonlinear partial differential equations. This method was proven to be very accurate compared to other numerical methods [8].

5.1.1. The Time-bandwidth product
As aforementioned, TBP is a very important metric to take into consideration when designing the optical fiber. However, another critical metric emerges as well, namely the distance-bandwidth product (DBWP). The TBP and the DBWP together make a crucial design parameter relating the temporal width, spectral width, and the length of the fiber.

The DBWP is simply the product of the fiber length and the input signal bandwidth, it is a characteristic of the fiber. The TBP and the DBWP show some of the trade-offs that are present in this design problem such as, for a given data rate, there are some constraints on the bandwidth, and the temporal-width of the input pulse, also the length of the fiber. Moreover, these constraints when violated produce performance degradation such as inter-symbol interference and high bit error rate.

The TBP was calculated along the distance of the fiber for the first seven derivatives of the Gaussian pulse. Fig. 5 illustrates the amount of change that the TBP of the pulse undergoes during propagation. It is assumed that the transmitted pulses are transform-limited pulses, having the minimum TBP for the given pulse. It is worth mentioning that transform-limited pulses can minimize the effect of chromatic dispersion during propagation in the fiber. Thus, maximizing the possible transmission distance. It can be deduced from Fig. 5 that the 1st-order derivative of the Gaussian pulse maintains the best performance along the fiber regarding its TBP, this observation results in a design trade-off as discussed in the following sub-section.

5.1.2. FCC Violation ratio
FCC VR was also calculated along the distance of the fiber for the first seven derivatives of the Gaussian pulse. As shown in Fig. 6, the seven derivatives of the Gaussian pulse having near-constant FCC VR along the transmission distance. The non-linearity in the SMF is the cause for the slight change in the FCC VR along propagation distance. It can be also noted, as shown in Fig. 6, that the 6th- and the 4th-order derivatives of the Gaussian pulse have the best FCC VR along the channel length, while the 1st-order derivative has the worst FCC VR.

Fig. 5 and 6 illustrate the evident trade-off between the quality of the transmitted UWB pulse, and its compliance to the FCC constraints. It can be noted that the 2nd-order derivative of the Gaussian pulse maintains great performance regarding both metrics, TBP and the FCC VR.

6. Conclusion
UWB technology has various design trade-offs and parameters; the most important is obeying the FCC regulations and constraints while maintaining the quality of the transmitted pulses. The analysis carried out, in this paper, for a complete optical communications system has proven the significance of the TBP and the FCC VR in the design process of a similar system, the TBP quantifies the quality of the signal throughout the system by measuring distortion in the pulse shape, whilst the FCC VR measures the compliance of the pulses to the FCC constraints. The mathematical closed form obtained for the TBP provides an accurate representation of the quality of UWB pulses in a similar system. The analysis provided in this paper gives great insight regarding the present trade-offs in the
design of the generated UWB pulses, and it was shown that the 2nd-order derivative of the Gaussian pulse maintains great performance. Moreover, this mathematical analysis carried out for the first seven derivatives of the Gaussian and sech-shaped pulses is suitable for analytical synthesis and tracing of some optical devices such as the optical differentiator [9].

Acknowledgements

This work was supported by the Egyptian National Telecom Regulatory Authority (NTRA) Program CFP # 04.

References


